Minimum Sizes of Identifying Codes in Graphs Differing by One Vertex (OH2)

Charon, Honkala, Hudry, Lobstein (2013), *Cryptography and Communications* | Poster author: Vielzeuf Charles

Context and Motivation

Identifying codes model the placement of sensors on certain vertices of a graph to **identify** (or locate) an unknown vertex from the sensors that "see" it.

Example application: Museum rooms
Consider a museum with rooms connected by corridors. We want to place detectors (sensors) in some rooms so that if an alarm is triggered, we can identify exactly which room it came from based on which detectors detect it. The detectors have a detection radius r: they can detect alarms in rooms within distance r. The goal is to minimize the number of detectors needed while ensuring every room can be uniquely identified. We note by $\gamma_r(G)$ **the minimum number of detectors needed to identify all rooms in a graph** G.

Article OH2 studies the **stability** of the parameter $\gamma_r(G)$ when modifying the graph by **adding or deleting a vertex**. With G^* obtained from G by adding/deleting a vertex, provided that G and G^* remain **identifiable**, we compare:

$$\gamma_r(G^*) - \gamma_r(G) \quad \text{and} \quad \frac{\gamma_r(G^*)}{\gamma_r(G)}.$$

Main Definitions

Let $G = (V, E)$ be a simple undirected graph. For $v \in V$ and $r \geq 1$, the **ball** of radius r is $B_{G,r}(v) = \{x \in V : d_G(v, x) \leq r\}$.

Two vertices $x \neq y$ are (G, S, r)-**twins** if

$$B_{G,r}(x) \cap S = B_{G,r}(y) \cap S.$$

The graph is r-**twin-free** if no such twins exist (when $S = V$).

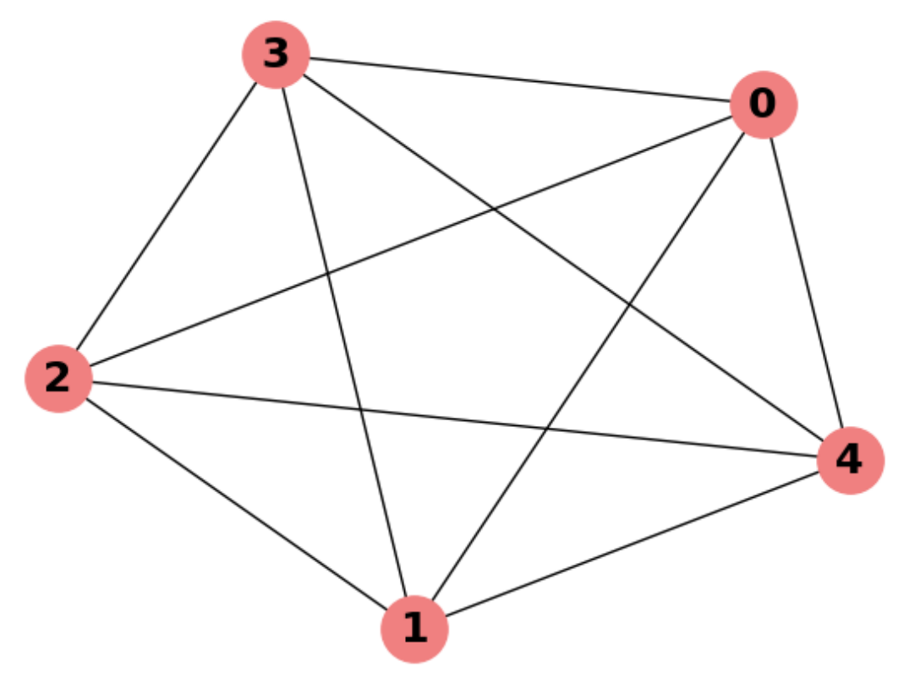
A **code** is a subset $\mathcal{C} \subseteq V$; the elements of C are called **codewords**.
For a vertex v, the associated **identifying set** is:

$$I_{G,\mathcal{C},r}(v) = B_{G,r}(v) \cap \mathcal{C}.$$

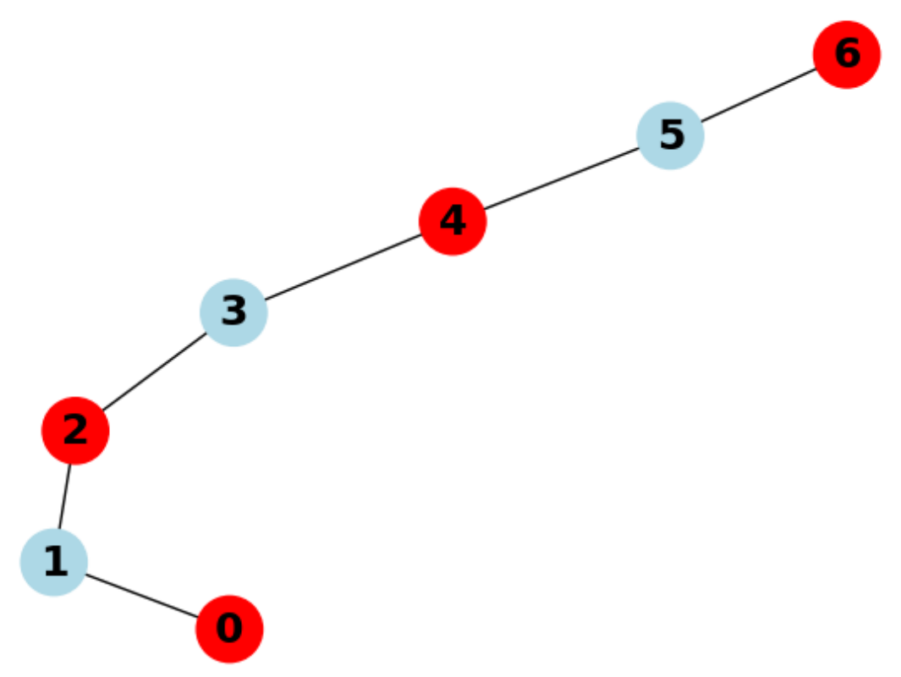
The code $\mathcal{C}$ is r-**identifying** if the sets $I_{G,\mathcal{C},r}(v)$, $v \in V$, are:

- **nonempty** (every vertex is covered),
- **all distinct** (every pair of vertices is separated).

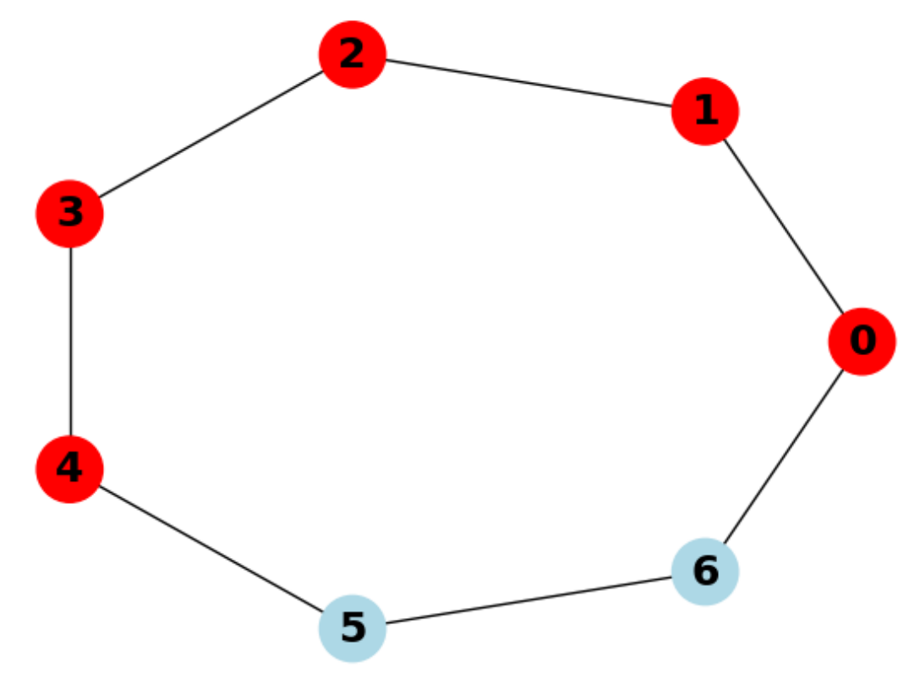
A graph admits an r-identifying code only if it is r-twin-free: two r-twins would have the same identifying set $I_{G,\mathcal{C},r}(\cdot)$ for any code $\mathcal{C}$, so they could not be separated.
When G is r-twin-free, we define $\gamma_r(G) = \min\{|\mathcal{C}| : \mathcal{C}$ is an r-identifying code of $G\}$.



Complete graph K_5
(not r-twin-free)



Path P_7
$\gamma_1(P_7) = 4$, Code: [0, 2, 4, 6]



Cycle C_7
$\gamma_1(C_7) = 5$, Code: [0, 1, 2, 3, 4]

Main Questions

Given an r-twin-free graph G, and G^* obtained by **adding** or **deleting** a vertex, if G^* is still r-twin-free:

- What differences are possible for $\gamma_r(G^*) - \gamma_r(G)$?
- What ratios are possible for $\gamma_r(G^*)/\gamma_r(G)$?

OH2 provides **extremal constructions** showing that these quantities can vary strongly, including for **connected** graphs.

Lemmas

Lemma 1 [(G, S, r)-twin transitivity] In a graph $G = (V, E)$, if x, y, z are three distinct vertices, if S is a subset of V, if x and y are (G, S, r)-twins and if y and z are (G, S, r)-twins, then x and z are (G, S, r)-twins.

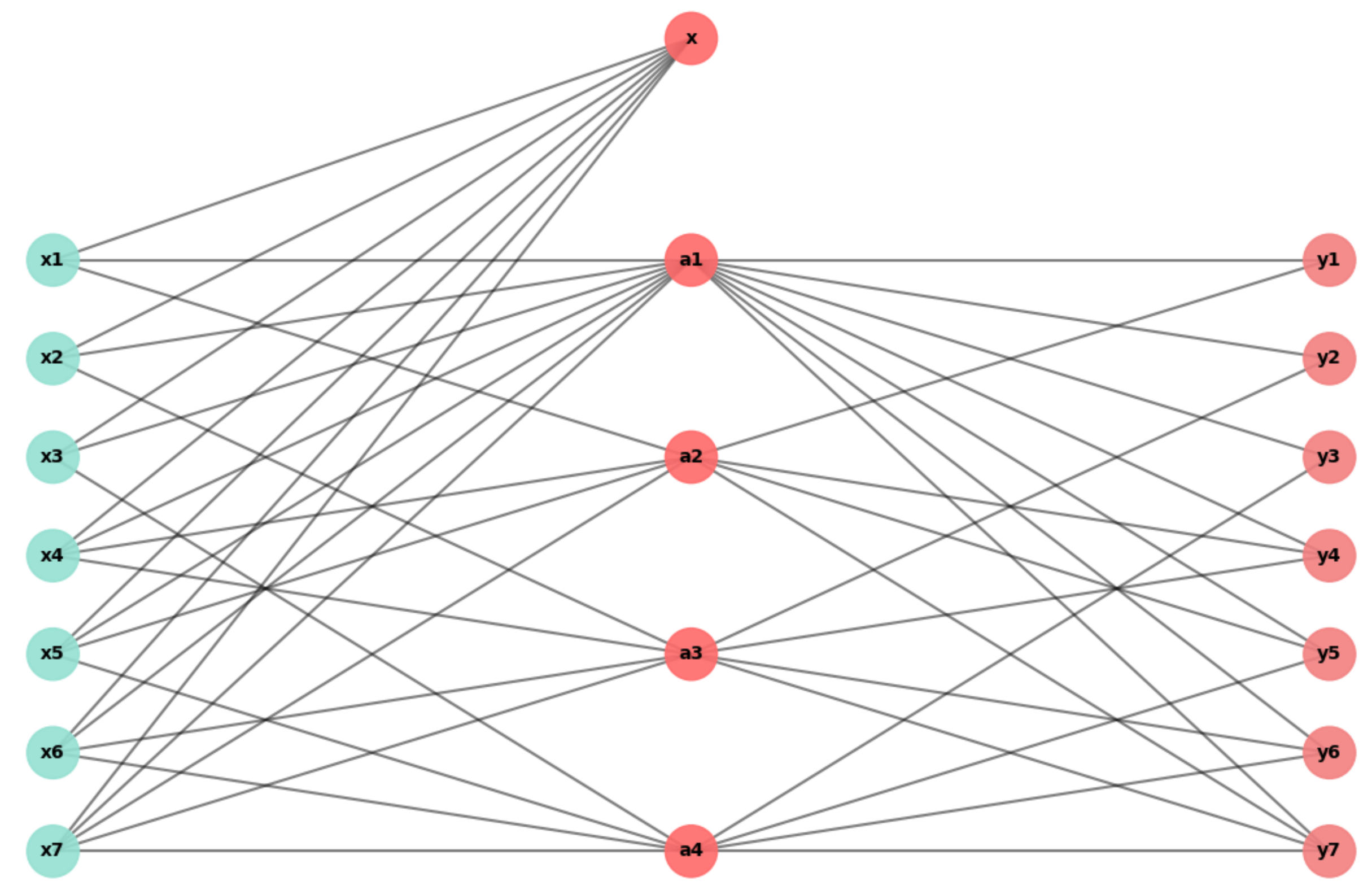
Lemma 2 If C is an r-identifying code in a graph $G = (V, E)$, then so is any set S such that $C \subseteq S \subseteq V$.

Case $r = 1$

Let $k \geq 1$ be an arbitrary integer. There exist two (connected) 1-twin-free graphs G and G_x, where G_x denotes the graph obtained from G by deleting the vertex x and G has $2k + \lceil \log_2(k + 1) \rceil + 2$ vertices, such that

$$\gamma_1(G) \leq \lceil \log_2(k + 1) \rceil + 2 \quad \text{and} \quad \gamma_1(G_x) \geq k.$$

Proof: Construction with two similar sets of x- and y-vertices, each identifiable via the codewords a, and a vertex x adjacent to only one of these sets.



Example $k = 7$: the codewords $a_i + x$ suffice to identify all x_i and y_i in G. x is the vertex removed to get G_x;

Open question: Can we do better?

Let $G = (V, E)$ be any 1-twin-free graph with at least three vertices. For any vertex $x \in V$ such that G_x is 1-twin-free, we have:

$$\gamma_1(G_x) \geq \gamma_1(G) - 1.$$

Proof: From a minimum 1-identifying code in G_x, one builds a 1-identifying code in G of size at most $\gamma_1(G_x) + 1$ by adding x if it is not 1-covered, or (when x is 1-covered but $C - \{x\}$ fails to 1-separate x from some vertex v) by adding a vertex z that 1-covers exactly one of x and v (such z exists because G is 1-twin-free); hence $\gamma_1(G) \leq \gamma_1(G_x) + 1$.

Corollary: If $\gamma_1(G_x) \leq a$ and $\gamma_1(G) \geq a + 1$, then $\gamma_1(G_x) = a$ and $\gamma_1(G) = a + 1$.

Case $r \geq 2$

For $r \geq 2$, both $\gamma_r(G_x) - \gamma_r(G)$ and $\gamma_r(G_x)/\gamma_r(G)$ can be **arbitrarily large or small**.

General view. The paper uses cycles (e.g. $\gamma_r(C_n) = n/2$ for even $n \geq 2r + 4$), then connected graphs with r *even* and r *odd* where the difference grows with n; path-based constructions yield ratios that can be made arbitrarily large; other families give exact γ_r for graphs of order $pr + 1$ and $pr + 2$. Both the difference and the ratio can be made arbitrarily large or arbitrarily small (see best know ratios from conclusion).

General Conclusion

Table 1 summarizes the results of the paper: possible ranges for the difference $\gamma_r(G_x) - \gamma_r(G)$ and the ratio $\gamma_r(G_x)/\gamma_r(G)$ when G_x is obtained from G by adding or deleting one vertex. The asymptotic behaviour is given for n large with respect to r; $\approx$ denotes approximate worth known growth.

r	r	comment	$\gamma_r(G_x) - \gamma_r(G)$	$\gamma_r(G_x)/\gamma_r(G)$
$r = 1$		always	≥ -1	
$r = 1$		(connected) graphs	$\gtrsim 0.5n - 1.5 \log_2 n$	$\gtrsim n/(2 \log_2 n)$
≥ 2	even	connected graphs	$\gtrsim n/4$	
≥ 2	odd	connected graphs	$\gtrsim n(3r - 1)/(12r)$	
≥ 2	any	not connected graphs	$\gtrsim n(2r - 2)/(2r + 1)$	
≥ 2	any	(connected) graphs		$\gtrsim n/(2r^2 \log_2 n)$
≥ 2	any	(connected) graphs	$\lesssim -n(r - 1)/r$	$\lesssim r(r + 1) \log_2 n/n$

Table 1: The difference $\gamma_r(G_x) - \gamma_r(G)$ and ratio $\gamma_r(G_x)/\gamma_r(G)$, as functions of n and r.

Sources

More results

- A. Lobstein, O. Hudry, I. Charon, « Locating-domination and identification », Ch. 6 in *Topics in Domination in Graphs*, eds. T. Haynes, S. Hedetniemi, M. Henning, Springer, 2020, pp. 251–299.
- O. Hudry, V. Junnila, A. Lobstein, « On Iiro Honkala's contributions to identifying codes », *Fundamenta Informaticae* **191** (3–4), 2024, pp. 165–196.

Applications of identifying codes

- K. Basu, A. Sen, « Identifying individuals associated with organized criminal networks: A social network analysis », *Social Networks* **64**, 2021, pp. 42–54.
- K. Basu, M. Padhee, S. Roy, A. Pal, A. Sen, M. Rhodes, B. Keel, « Health Monitoring of Critical Power System Equipments using Identifying Codes », preprint, 2018.
- S. Sengupta, K. Basu, A. Sen, S. Kambhampati, « Moving Target Defense for Robust Monitoring of Electric Grid Transformers in Adversarial Environments », *GameSec* 2020. Uses **discriminating codes** (variant of identifying codes).